

# A NEW FAMILY OF $q$ -ANALOGUE OF GENOCCHI NUMBERS AND POLYNOMIALS OF HIGHER ORDER

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ABSTRACT. The new  $q$ -Euler polynomials was introduced by T. Kim in “ $q$ -Generalized Euler numbers and polynomials, Russian Journal of Mathematical Physics, Vol. 13, No. 3, 2006, pp. 293-308” by means of the following generating function:

$$\sum_{j=0}^{\infty} \frac{z^j}{[j]_q!} E_{j,q}(x) = \frac{[2]_q}{e_q(z) + 1} e_q(xz).$$

In this work, we consider the generating function of Kim’s  $q$ -Euler polynomials and introduce new generalization of  $q$ -Genocchi polynomials and numbers of higher order. Also, we give surprising identities for studying in Analytic Numbers Theory and especially in Mathematical Physics. Moreover, by applying  $q$ -Mellin transformation to generating function of  $q$ -Genocchi polynomials of higher order and so we define  $q$ -Hurwitz-Zeta type function which interpolates of this polynomials at negative integers.

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## 1. Introduction

Throughout this work, we assume that  $q \in \mathbb{C}$  with  $|q| < 1$ . The  $q$ -integer of  $x$  is defined by  $[x]_q = \frac{1-q^x}{1-q}$  and note that  $\lim_{q \rightarrow 1} [x]_q = x$ . The  $q$ -derivative is defined by F. H. Jackson as follows:

$$(1) \quad D_q f(x) = \frac{d}{d_q x} f(x) = \frac{f(x) - f(qx)}{(1-q)x}.$$

Taking  $f(x) = x^n$  in (1), it becomes as follows:

$$D_q x^n = \frac{x^n - (qx)^n}{(1-q)x} = [n]_q x^{n-1} \text{ and } \left( \frac{d}{d_q x} \right)^n f(x) = [n]_q!$$

where  $[n]_q! = [n]_q [n-1]_q \cdots [1]_q$ . Now, we give definitions of two kinds of  $q$ -exponential functions as follows:

For any  $z \in \mathbb{C}$  with  $|z| < 1$ ,

$$(2) \quad e_q(z) = \sum_{l=0}^{\infty} \frac{z^l}{[l]_q!} \text{ and } E_q(z) = \sum_{l=0}^{\infty} q^{\binom{l}{2}} \frac{z^l}{[l]_q!}.$$

By (2), it is not difficult to show that  $[l]_{\frac{1}{q}}! = q^{-\binom{l}{2}} [l]_q!$ . Then, we have the following

$$(3) \quad e_{\frac{1}{q}}(z) = E_q(z).$$

For the  $q$ -commuting variables  $x$  and  $y$  such that  $yx = qxy$ , we know that

$$(4) \quad e_q(x + y) = e_q(x) e_q(y).$$

The  $q$ -integral was defined by Jackson as follows:

$$(5) \quad \int_0^x f(\xi) d_q \xi = (1 - q) x \sum_{l=0}^{\infty} f(q^l x) q^l$$

provided that the series on the right hand side converges absolutely.

In particular, if  $f(\xi) = \xi^n$ , then we have

$$(6) \quad \int_0^x \xi^n d_q \xi = \frac{1}{[n+1]_q} x^{n+1}.$$

The definitions of  $q$ -integral and  $q$ -derivative imply the following formula:

$$(7) \quad D_q \left( \int_0^x f(t) d_q t \right) = f(x)$$

and

$$(8) \quad D_q(f(x)g(x)) = f(x)D_q(g(x)) + g(qx)D_q(f(x)).$$

For more informations of Eqs. (1-8), you can refer to [19-23].

The ordinary Euler numbers and polynomials are defined via the following generating function:

$$e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = \frac{2}{e^t + 1} e^{xt}, \quad |t| < \pi$$

where the usual convention about replacing  $E^n(x)$  by  $E_n(x)$  (see [1], [2], [3], [5], [6], [13]).

In [6], the new  $q$ -generalization of Euler polynomials are introduced by T. Kim as follows:

$$\sum_{j=0}^{\infty} \frac{z^j}{[j]_q!} E_{j,q}(x) = \frac{[2]_q}{e_q(z) + 1} e_q(xz).$$

By using the above generating function, Kim gave some interesting and fascinating properties for new  $q$ -generalization of Euler numbers and polynomials. We note that these polynomials are used to study in Analytic Numbers Theory. So, in the next section, we shall introduce generating function of  $q$ -Genocchi numbers and polynomials of higher order. Additionally, we shall give their applications.

## 2. New $q$ -Genocchi numbers and polynomials of higher order

In this section, we introduce generating function for  $q$ -Genocchi polynomials of higher order by using Kim's method in [6]. Thus, we now start as follows:

$$(9) \quad S_q(t : \alpha) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} G_{n,q}^{(\alpha)}.$$

Here  $G_{n,q}^{(\alpha)}$  is called as the  $q$ -Genocchi numbers of higher order. By using  $q$ -derivative operator, we compute as follows:

$$(10) \quad S_q\left(\frac{d}{d_q x} : \alpha\right) x^k = \sum_{n=0}^{\infty} \frac{1}{[n]_q!} G_{n,q}^{(\alpha)} \left(\frac{d}{d_q x}\right)^n x^k = \sum_{n=0}^k \binom{k}{n}_q G_{n,q}^{(\alpha)} x^{k-n},$$

where

$$\binom{k}{n}_q = \frac{[k]_q [k-1]_q \cdots [k-n+1]_q}{[n]_q!}.$$

Similarly, by (10), we develop as follows:

$$\begin{aligned} S_q\left(\frac{d}{d_q x} : \alpha\right) e_q(tx) &= \sum_{j=0}^{\infty} \frac{G_{j,q}^{(\alpha)}}{[j]_q!} \left(\frac{d}{d_q x}\right)^j \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!} t^k \\ &= \sum_{j=0}^{\infty} \frac{t^j}{[j]_q!} G_{j,q}^{(\alpha)}(x) \\ &= S_q(x, t : \alpha). \end{aligned}$$

From this point of view, we can also consider the  $q$ -Genocchi polynomials of higher order in the form:

$$(11) \quad \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} G_{n,q}^{(\alpha)}(x) = \left(\frac{[2]_q z}{e_q(z) + 1}\right)^{\alpha} e_q(zx).$$

As  $q \rightarrow 1$  and  $\alpha = 1$  in Eq. (11), we easily reach the following

$$\lim_{q \rightarrow 1} G_{n,q}^{(1)}(x) = G_n(x)$$

which  $G_n(x)$  is known as ordinary Genocchi polynomials (for details, see [5], [8], [14], [15], [17], [18]).

By (9) and (11), we readily see that

$$\sum_{j=0}^{\infty} \frac{z^j}{[j]_q!} G_{j,q}^{(\alpha)}(x) = \sum_{j=0}^{\infty} \left( \sum_{n=0}^j \binom{j}{n}_q x^{j-n} G_{n,q}^{(\alpha)} \right) \frac{z^j}{[j]_q!}.$$

By comparing the coefficients of  $\frac{z^j}{[j]_q!}$  on both sides of the above equation, then we obtain the following theorem.

**Theorem 2.1.** *For any  $j \in \mathbb{N}$ , we have*

$$G_{j,q}^{(\alpha)}(x) = \sum_{n=0}^j \binom{j}{n}_q x^{j-n} G_{n,q}^{(\alpha)}.$$

By applying  $q$ -derivative operator to (11), then we see that

$$\sum_{n=1}^{\infty} \frac{z^n}{[n]_q!} \left\{ \frac{d}{d_q x} G_{n,q}^{(\alpha)}(x) \right\} = z \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} G_{n,q}^{(\alpha)}(x).$$

By comparing the coefficients of  $z^n$  on both sides of the above equation, we arrive the following theorem.

**Theorem 2.2.** *For any  $n \in \mathbb{N}^* = \{0, 1, 2, 3, \dots\}$ , we have*

$$\frac{d}{d_q x} G_{n,q}^{(\alpha)}(x) = [n]_q G_{n-1,q}^{(\alpha)}(x).$$

For  $q$ -commuting variables  $x$  and  $y$  ( $yx = qxy$ ), we note that

$$\begin{aligned} \sum_{l=0}^{\infty} \frac{z^l}{[l]_q!} G_{l,q}^{(\alpha)}(x+y) &= \left( \frac{[2]_q z}{e_q(z) + 1} \right)^{\alpha} e_q(z(x+y)) \\ &= e_q(zy) \left( e_q(zx) \left( \frac{[2]_q z}{e_q(z) + 1} \right)^{\alpha} \right) \\ &= \sum_{l=0}^{\infty} \left( \sum_{j=0}^l \binom{l}{j}_q y^{l-j} G_{j,q}^{(\alpha)}(x) \right) \frac{z^l}{[l]_q!}. \end{aligned}$$

As a result, we procure the following theorem.

**Theorem 2.3.** *For any  $n \in \mathbb{N}^*$ , we have*

$$G_{n,q}^{(\alpha)}(x+y) = \sum_{j=0}^n \binom{n}{j}_q y^{n-j} G_{j,q}^{(\alpha)}(x).$$

By expression (11), we compute as follows:

$$\begin{aligned} (12) \quad & \sum_{l=0}^{\infty} \frac{z^l}{[l]_q!} G_{l,q}^{(\alpha+\beta)}(x) \\ &= \left[ \left( \frac{[2]_q z}{e_q(z) + 1} \right)^{\alpha} \right] \left[ \left( \frac{[2]_q z}{e_q(z) + 1} \right)^{\beta} e_q(zx) \right] \\ &= \left[ \sum_{j=0}^{\infty} \frac{z^j}{[j]_q!} G_{j,q}^{(\alpha)} \right] \left[ \sum_{k=0}^{\infty} \frac{z^k}{[k]_q!} G_{k,q}^{(\beta)}(x) \right] \end{aligned}$$

by using Cauchy product on the above equation, we derive that

$$(13) \quad \sum_{l=0}^{\infty} \frac{z^l}{[l]_q!} \left( \sum_{n=0}^l \binom{l}{n}_q G_{n,q}^{(\alpha)} G_{l-n,q}^{(\beta)}(x) \right).$$

Comparing the coefficients of Eqs. (12) and (13), then we present the following theorem.

**Theorem 2.4.** *For  $l \in \mathbb{N}^*$ , then we have*

$$G_{l,q}^{(\alpha+\beta)}(x) = \sum_{n=0}^l \binom{l}{n}_q G_{n,q}^{(\alpha)} G_{l-n,q}^{(\beta)}(x).$$

Jackson are defined the  $q$ -analogue of the Gamma function by

$$(14) \quad \Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1-q)^{1-x}, \quad x \neq 0, -1, -2, \dots$$

which have the following properties:

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x), \quad \Gamma_q(1) = 1 \quad \text{and} \quad \lim_{q \rightarrow 1^-} \Gamma_q(x) = \Gamma(x), \quad \Re(x) > 0.$$

It has the  $q$ -integral representation as follows:

$$(15) \quad \Gamma_q(s) = \int_0^{\frac{1}{1-q}} t^{s-1} E_q(-qt) d_q t.$$

When  $\frac{\log(1-q)}{\log q} \in \mathbb{Z}$ , becomes

$$(16) \quad \Gamma_q(s) = \int_0^\infty t^{s-1} E_q(-qt) d_q t.$$

The  $q$ -Mellin transformation of a suitable function  $f$  on  $\mathbb{R}_{q,+}$  is defined by

$$(17) \quad M_q(f)(s) = \int_0^\infty t^{s-1} f(t) d_q t$$

(for details of Eqs. (14-17), see [10], [20], [21]).

In [23], the novel  $q$ -differential operator was defined by Rubin as follows:

$$(18) \quad \partial_q(f)(x) = \frac{f(q^{-1}x) + f(-q^{-1}x) - f(qx) + f(-qx) - 2f(-x)}{2(1-q)x}.$$

By (18), we note that

$$(19) \quad \lim_{q \rightarrow 1} \partial_q(f)(x) = f'(x).$$

By applying Rubin's  $q$ -differential operator to the generating function of  $q$ -Genocchi numbers and polynomials of higher order, we compute as follows:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} \partial_q G_{n,q}^{(\alpha)}(x) &= \partial_q \left\{ \left( \frac{[2]_q z}{e_q(z) + 1} \right)^\alpha e_q(xz) \right\} \\ &= \sum_{n=0}^{\infty} \left( \frac{1}{2(1-q)} \sum_{l=0}^n \binom{n}{l}_q \left\{ \begin{array}{l} q^{-l} + (-1)^l q^{-l} - q^l \\ + (-1)^l q^l + 2(-1)^l \end{array} \right\} x^{l-1} G_{n-l,q}^{(\alpha)} \right) \frac{z^n}{[n]_q!}. \end{aligned}$$

By comparing the coefficients of  $\frac{z^n}{[n]_q!}$  on both sides of the above equation. Then, we state the following theorem.

**Theorem 2.5.** *Let  $T_q(l) = q^{-l} + (-1)^l q^{-l} - q^l + (-1)^l q^l + 2(-1)^l$ , then we get*

$$(20) \quad \partial_q G_{n,q}^{(\alpha)}(x) = \frac{1}{2(1-q)} \sum_{l=0}^n \binom{n}{l}_q T_q(l) x^{l-1} G_{n-l,q}^{(\alpha)}.$$

By (20), we readily derive the following

$$\begin{aligned} \partial_q G_{n,q}^{(\alpha)}(x) &= \frac{1}{(1-q)} \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2l}_q \{q^{-2l} + 1\} G_{n-2l,q}^{(\alpha)} \\ &\quad + \frac{1}{(q-1)} \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2l+1}_q \{q^{2l+1} + 1\} G_{n-1-2l,q}^{(\alpha)}. \end{aligned}$$

Here  $\lfloor \cdot \rfloor$  is Gauss' symbol. Consequently, we derive the following theorem.

**Theorem 2.6.** *For any  $n \in \mathbb{N}^*$ , we have*

$$\begin{aligned} (21) \quad \partial_q G_{n,q}^{(\alpha)}(x) &= \frac{1}{(1-q)} \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2l}_q \{q^{-2l} + 1\} G_{n-2l,q}^{(\alpha)} \\ &\quad + \frac{1}{(1-q)} \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2l+1}_q \{q^{2l+1} + 1\} G_{n-1-2l,q}^{(\alpha)}. \end{aligned}$$

From (20) and (21), we conclude as follows:

**Corollary 2.7.** *The following identity*

$$\begin{aligned} \sum_{l=0}^n \binom{n}{l}_q T_q(l) x^{l-1} G_{n-l,q}^{(\alpha)} &= \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2l}_q \{2q^{-2l} + 2\} G_{n-2l,q}^{(\alpha)} \\ &\quad + \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2l+1}_q \{2q^{2l+1} + 2\} G_{n-1-2l,q}^{(\alpha)} \end{aligned}$$

is true.

By (20), we have the following corollary.

**Corollary 2.8.** *For any  $n \in \mathbb{N}^*$ , we get*

$$\lim_{q \rightarrow 1} \partial_q G_{n,q}^{(\alpha)}(x) = n G_{n-1}^{(\alpha)}(x)$$

where  $G_n^{(\alpha)}(x)$  are called Genocchi numbers and polynomials of higher order which are defined by the following generating function:

$$\sum_{n=0}^{\infty} G_n^{(\alpha)}(x) \frac{t^n}{n!} = \left( \frac{2t}{e^t + 1} \right)^\alpha e^{xt} \text{ (see [5], [14]).}$$

We now give a  $q$ -analogue of D. Milićić's Lemma (see [22, page 1, Lemma 1.2.1]).

**Lemma 2.9.** *Let  $a_{n,q}$ ,  $n \in \mathbb{N}^* := \{0, 1, 2, 3, \dots\}$ , be complex numbers such that  $\sum_{n=0}^{\infty} |a_{n,q}|$  converges. Let*

$$\lambda = \{-n \mid n \in \mathbb{N}^* \text{ and } a_{n,q} \neq 0\}.$$

Then,

$$g_q(z) = \sum_{n=0}^{\infty} \frac{a_{n,q}}{[z+n]_q}$$

converges absolutely for  $z \in \mathbb{C} - \lambda$  and uniformly on bounded subsets of  $\mathbb{C} - \lambda$ . The  $q$ -function is a meromorphic function on complex plane with simple poles at the points in  $\lambda$  and  $\text{Res}(g_q, -n) = a_{n,q}$  for any  $-n \in \lambda$ .

*Proof.* By using similar method in lecture notes of D. Milić in [22], it is clear that if  $|[z]_q| < R$ , we see

$$|[z+n]_q| = |[z]_q + q^{zn}[n]_q| \geq |[n]_q - R|$$

for all  $\frac{1-q^n}{1-q} > R$ . Then, we get

$$\left| \frac{1}{[z+n]_q} \right| \leq \frac{1}{[n]_q - R}$$

for  $|[z]_q| < R$  and  $[n]_q > R$ . It follows that  $[n_0]_q > R$ , we have

$$\left| \sum_{n=n_0}^{\infty} \frac{a_{n,q}}{[z+n]_q} \right| \leq \sum_{n=0}^{\infty} \frac{|a_{n,q}|}{|[z+n]_q|} \leq \sum_{n=0}^{\infty} \frac{|a_{n,q}|}{[n]_q - R} \leq \frac{1}{[n_0]_q - R} \sum_{n=n_0}^{\infty} |a_{n,q}|$$

Hence, the series  $\sum_{n>R} \frac{a_{n,q}}{[z+n]_q}$  converges absolutely and uniformly on the disk  $\{z \mid |z| < R\}$  and defines there a meromorphic function. It follows that

$$\sum_{n=0}^{\infty} \frac{a_{n,q}}{[z+n]_q}$$

is a meromorphic function on that disk with the simple poles at the points of  $\lambda$  in  $\{z \mid |z| < R\}$ . Then, for any  $-n \in \lambda$ , we have

$$g_q(z) = \frac{a_{n,q}}{[z+n]_q} + \sum_{-j \in \lambda - \{n\}}^{\infty} \frac{a_{j,q}}{[z+j]_q} = \frac{a_{n,q}}{[z+n]_q} + \vartheta(z)$$

where  $\vartheta(z)$  is holomorphic at  $-n$ . This also shows that

$$\text{Res}(g_q, -n) = a_{n,q}.$$

Thus, we successfully complete the proof of lemma.  $\square$

We now want to indicate that  $\Gamma$  extends to a meromorphic function by using Lemma 2. 9. That is, we discover the following

$$\Gamma_q(z) = \int_0^{\infty} t^{z-1} E_q(-qt) d_q t = \int_0^1 t^{z-1} E_q(-qt) d_q t + \int_1^{\infty} t^{z-1} E_q(-qt) d_q t.$$

Then, the second integral converges for any complex  $z$  and represents an entire function. On the other hand, the  $q$ -exponential function is entire, and

we have

$$\begin{aligned}
\int_0^1 t^{z-1} E_q(-qt) d_q t &= \int_0^1 t^{z-1} \left\{ \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j+1}{2}}}{[j]_q!} t^j \right\} d_q t \\
&= \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j+1}{2}}}{[j]_q!} \left\{ \int_0^1 t^{z+j-1} d_q t \right\} \\
&= \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j+1}{2}}}{[j]_q!} \frac{1}{[z+j]_q}
\end{aligned}$$

for any  $z \in \mathbb{C}$ . Now also, we can write as follows:

$$\Gamma_q(z) = \int_1^{\infty} t^{z-1} E_q(-qt) d_q t + \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j+1}{2}}}{[j]_q!} \frac{1}{[z+j]_q}$$

for any  $z$  in the right half plane. From Lemma 2. 9, the right hand-side of the above identity defines a meromorphic function on the complex plane with simple poles at  $z = -j$ ,  $j \in \mathbb{N}^*$ . Then, we have the following theorem.

**Theorem 2.10.** *For any  $j \in \mathbb{N}^*$ , we derive the following*

$$(22) \quad \text{Res}(\Gamma_q, -j) = \frac{(-1)^j q^{\binom{j+1}{2}}}{[j]_q!}.$$

As  $q \rightarrow 1$  into (22), we easily derive that

$$\lim_{q \rightarrow 1} \text{Res}(\Gamma_q, -j) = \frac{(-1)^j}{j!}$$

which it is residue of Euler's Gamma function (see [22]).

Now also, by applying  $q$ -Mellin Transformation to generating function of  $q$ -Genocchi polynomials of higher order, then we compute as follows:

$$\begin{aligned}
\mathfrak{S}_q(z, x : \alpha) &= \frac{1}{\Gamma_{\frac{1}{q}}(z)} \int_0^{\infty} t^{z-\alpha-1} \{(-1)^{\alpha} S_q(x, -t : \alpha)\} d_{\frac{1}{q}} t \\
&= \sum_{l_1, l_2, \dots, l_{\alpha}=0}^{\infty} (-1)^{l_1+l_2+\dots+l_{\alpha}} \left\{ \frac{1}{\Gamma_{\frac{1}{q}}(z)} \int_0^{\infty} t^{z-1} E_{\frac{1}{q}} \left( -\frac{t}{q} \left( qx + q \sum_{k=1}^{\alpha} l_k \right) \right) d_{\frac{1}{q}} t \right\} \\
&= [2]_q^{\alpha} \sum_{l_1, l_2, \dots, l_{\alpha}=0}^{\infty} \frac{q^{-z} (-1)^{l_1+l_2+\dots+l_{\alpha}}}{(l_1 + l_2 + \dots + l_{\alpha} + x)^z}.
\end{aligned}$$

So, we now introduce definition of  $q$ -Hurwitz-Zeta type function as follows:

**Definition 1.** *For any  $z \in \mathbb{C}$ , then we define*

$$\mathfrak{S}_q(z, x : \alpha) = [2]_q^{\alpha} \sum_{l_1, l_2, \dots, l_{\alpha}=0}^{\infty} \frac{(-1)^{l_1+l_2+\dots+l_{\alpha}}}{(qx + q \sum_{k=1}^{\alpha} l_k)^z}.$$

Via the above definition, we derive interpolation functions for  $q$ -Genocchi numbers and polynomials of higher order at negative integers with the following theorem.



**Theorem 2.11.** *The following equality holds:*

$$\mathfrak{S}_q(-n, x : \alpha) = \frac{q^{-n} G_{n+\alpha, q}^{(\alpha)}(x)}{[\alpha]_q! \binom{n+\alpha}{\alpha}_q}.$$

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